

# PERSISTENCE OF FIXED POINTS UNDER RIGID PERTURBATIONS OF MAPS

SALVADOR ADDAS-ZANATA AND PEDRO A. S. SALOMÃO

**ABSTRACT.** Let  $f : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$  be a real-analytic annulus diffeomorphism which is homotopic to the identity map and preserves an area form. Assume that for some lift  $\tilde{f} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$  we have  $\text{Fix}(\tilde{f}) = \mathbb{R} \times \{0\}$  and that  $\tilde{f}$  positively translates points in  $\mathbb{R} \times \{1\}$ . Let  $\tilde{f}_\epsilon$  be the perturbation of  $\tilde{f}$  by the rigid horizontal translation  $(x, y) \mapsto (x + \epsilon, y)$ . We show that for all  $\epsilon > 0$  sufficiently small we have  $\text{Fix}(\tilde{f}_\epsilon) = \emptyset$ . The proof follows from Kerékjártó's construction of Brouwer lines for orientation preserving homeomorphisms of the plane with no fixed points. This result turns out to be sharp with respect to the regularity assumption: there exists a diffeomorphism  $f$  satisfying all the properties above, except that  $f$  is not real-analytic but only smooth, so that the above conclusion is false. Such a map is constructed via generating functions.

## 1. INTRODUCTION

Let us denote by  $\text{Diff}^k(\mathbb{D})$  the set of orientation and area preserving  $C^{k \geq 1}$ -diffeomorphisms  $\hat{h} : \mathbb{D} \rightarrow \mathbb{D}$ , defined in the closed disk  $\mathbb{D} := \{z \in \mathbb{R}^2 : |z| \leq 1\}$ , which fixes the origin  $0 \in \mathbb{D}$ . We denote by  $\text{Diff}_0^k(\mathbb{D}) \subset \text{Diff}^k(\mathbb{D})$  the subset of diffeomorphisms satisfying

$$\text{Fix}(\hat{h}) := \{\hat{h}(z) = z\} = \{0\} \text{ and } D\hat{h}(0) = Id.$$

Here we are considering the usual area form  $dz_1 \wedge dz_2$  on  $\mathbb{R}^2$  with coordinates  $(z_1, z_2)$ .

In this paper we address the following question:

**(Q1)** under what conditions can we find  $\hat{g} \in \text{Diff}^k(\mathbb{D})$  arbitrarily  $C^k$ -close to  $\hat{h}$  so that  $\text{Fix}(\hat{g}) = \{0\}$  and  $D\hat{g}(0) = e^{2\pi\epsilon i}$ ,  $\epsilon \in \mathbb{R} \setminus \mathbb{Q}$ ?

Before stating the main results we need some definitions.

**Definition 1.1.** (1) Let  $A := S^1 \times [0, 1]$  be the closed annulus, where  $S^1$  is identified with  $\mathbb{R}/\mathbb{Z}$ . Let  $\tilde{A} := \mathbb{R} \times [0, 1]$  be the infinite strip and  $p : \tilde{A} \rightarrow A$  be the covering map  $(x, y) \mapsto (x \bmod 1, y)$ .

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- (2) Let  $p_1 : \tilde{A} \rightarrow \mathbb{R}$  and  $p_2 : \tilde{A} \rightarrow \mathbb{R}$  be the projections of  $\tilde{A}$  into the first and second factors, respectively. We also denote by  $p_1$  and  $p_2$  the respective projections defined on  $A$ .
- (3) Let  $\text{Diff}^k(A)$  be the space of area preserving  $C^k$ -diffeomorphisms  $f : A \rightarrow A$ , where  $k \in \mathbb{N} \cup \{\infty, \omega\}$ , which are homotopic to the identity map. Let  $\text{Diff}_0^k(A) \subset \text{Diff}^k(A)$  denote the diffeomorphisms which satisfy the following conditions:  $f(x, 0) = (x, 0), \forall x \in S^1$  and if  $\tilde{f} : \tilde{A} \rightarrow \tilde{A}$  is the lift of  $f$  such that  $\tilde{f}(x, 0) = (x, 0), \forall x \in \mathbb{R}$ , then  $\text{Fix}(\tilde{f}) = \mathbb{R} \times \{0\}$ . Moreover, we require that

$$(1) \quad p_1 \circ \tilde{f}(x, 1) > x, \forall x \in \mathbb{R}.$$

- (4) Let  $\text{Diff}_0^k(\tilde{A})$  be the lifts of maps in  $\text{Diff}_0^k(A)$  which fix all points in  $\mathbb{R} \times \{0\}$ .

Now if  $\hat{h} \in \text{Diff}_0^k(\mathbb{D})$ , we obtain a map  $f := b^{-1} \circ \hat{h} \circ b$  induced by  $b : A \rightarrow \mathbb{D}$ , defined by

$$b(x, y) := (\sqrt{y} \cos 2\pi x, -\sqrt{y} \sin 2\pi x),$$

where  $(x, y)$  are coordinates in  $A$ . Notice that  $f$  preserves the area form  $dx \wedge dy$ . We assume that  $f$  extends to a map in  $\text{Diff}^k(A)$ . Clearly,  $S^1 \times \{0\}$  corresponds to the blow up of  $0 \in \mathbb{D}$  and  $S^1 \times \{1\}$  corresponds to  $\partial\mathbb{D}$ . Also, since  $\hat{h} \in \text{Diff}_0^k(\mathbb{D})$ , it follows that either  $f$  or  $f^{-1}$  admits a lift  $\tilde{f} \in \text{Diff}_0^k(\tilde{A})$ . In fact, either  $p_1 \circ \tilde{f}(x, 1) > x, \forall x \in \mathbb{R}$  or  $p_1 \circ \tilde{f}(x, 1) < x, \forall x \in \mathbb{R}$ . After possibly interchanging  $f$  with  $f^{-1}$  we may assume without loss of generality that (1) is satisfied.

Given  $\epsilon \in \mathbb{R}$  we consider the diffeomorphism

$$(2) \quad \tilde{f}_\epsilon : \tilde{A} \rightarrow \tilde{A} : (x, y) \mapsto \tilde{f}(x, y) + (\epsilon, 0).$$

The map  $\tilde{f}_\epsilon$  naturally induces a diffeomorphism  $f_\epsilon : A \rightarrow A$  given by

$$(3) \quad f_\epsilon = p \circ \tilde{f}_\epsilon \circ p^{-1}.$$

Notice that the translated map  $f_\epsilon$  corresponds to blowing up the map  $\hat{h} \in \text{Diff}_0^k(\mathbb{D})$  after compounding it with the rigid rotation  $z \mapsto e^{2\pi\epsilon i}z$ .

Our first result is the following theorem.

**Theorem 1.2.** *Let  $f \in \text{Diff}_0^\omega(A)$  and  $\tilde{f} \in \text{Diff}_0^\omega(\tilde{A})$  be a lift of  $f$ . Then there exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$ , we have  $\text{Fix}(\tilde{f}_\epsilon) = \emptyset$ .*

**Remark 1.3.** The hypothesis  $\tilde{f}(x, 0) = (x, 0), \forall x \in \mathbb{R}$  can be weakened to  $p_1 \circ \tilde{f}(x, 0) \geq x, \forall x \in \mathbb{R}$ , as is easily seen from the proof.

**Remark 1.4.** From the classical Poincaré-Birkhoff theorem  $\tilde{f}_\epsilon$  has fixed points in  $\text{interior}(\tilde{A})$  for all  $\epsilon < 0$  sufficiently small.

Our next result proves sharpness of the real-analyticity assumption in Theorem 1.2, i.e, this phenomenon does not occur assuming only smoothness.

**Theorem 1.5.** *There exist  $f \in \text{Diff}_0^\infty(A)$  and a sequence of positive real numbers  $\epsilon_n \rightarrow 0^+$  as  $n \rightarrow \infty$  such that  $\text{Fix}(\tilde{f}_{\epsilon_n}) \neq \emptyset$ , where  $\tilde{f} \in \text{Diff}_0^\infty(\tilde{A})$  is the special lift of  $f$  and  $\tilde{f}_{\epsilon_n}$  is defined as in (2), for all  $n \in \mathbb{N}$ .*

The proof of Theorem 1.2 strongly relies on a construction due to B. de Kerékjártó [3] of Brouwer lines for orientation preserving homeomorphisms of the plane which have no fixed point. Here, the hypothesis of real-analyticity of  $f$  plays an important role. We argue indirectly assuming the existence of a sequence  $\epsilon_n \rightarrow 0^+$  such that  $\tilde{f}_{\epsilon_n}$  admits a fixed point  $z_n$ . We can assume that  $z_n$  converges to a point  $\bar{z}$  at the lower boundary component of  $\tilde{A}$ . The real-analyticity hypothesis then allows one to conclude the existence of a small real analytic curve  $\gamma_0$  starting at  $\bar{z}$ , which is a graph in the vertical direction, so that  $\tilde{f}$  moves its point horizontally to the left. Since  $\tilde{f}$  has no fixed point in  $\text{interior}(\tilde{A})$ , the curve  $\gamma_0$  is then prolonged to a Brouwer line  $L \subset \tilde{A}$ , following Kerékjártó's construction. We analyse all possibilities for the behaviour of  $L$  and each of them yields a contradiction. Here, we strongly use the fact that  $\tilde{f}$  moves points in the upper boundary of  $\tilde{A}$  to the right.

The smooth map  $f$  in Theorem 1.5 is obtained from a special generating function on  $\tilde{A}$ . More precisely, first we define a diffeomorphism  $\psi : \tilde{A} \rightarrow \tilde{A}$  supported in the sequence of balls  $B_k \subset \tilde{A}$  centered at  $(0, 3/2^{k+2})$  and radius  $1/2^{k+3}$ , converging to the origin. Using the function  $h(t) = e^{-1/t}$ , which extends smoothly at  $t = 0$  as a flat point, we define the generating function by  $g(p) = h \circ p_2 \circ \psi(p)$ , where  $p_2$  is the projection in the vertical direction. The diffeomorphism associated to  $g$ , which is a priori defined only in a small neighbourhood of the origin, is then suitably re-scaled in order to find the diffeomorphism  $f$  of the annulus satisfying all the requirements.

As one can see,  $f$  satisfies all hypotheses of Theorem 1.2 except that it is not real-analytic at a unique point in the lower boundary. This follows from the flatness of  $h$  at  $t = 0$  and therefore the example in Theorem 1.5 shows the sharpness of the regularity assumption in Theorem 1.2.

## 2. KERÉKJÁRTÓ'S CONSTRUCTION OF BROUWER LINES

In this section we denote by  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  an orientation preserving homeomorphism of the plane satisfying

$$(4) \quad \text{Fix}(h) = \emptyset.$$

The following periodicity in  $x$  is assumed

$$(5) \quad h(x+1, y) = h(x, y) + (1, 0), \forall (x, y) \in \mathbb{R}^2.$$

**Definition 2.1.** (a) We call  $\alpha \subset \mathbb{R}^2$  a simple arc if  $\alpha$  is the image of a topological embedding  $\psi : [0, 1] \rightarrow \mathbb{R}^2$ . We may consider the parametrization  $\psi$  of the arc  $\alpha$ , which will also be denoted by  $\alpha$ . We also identify all the parameterizations of  $\alpha$  which are induced by orientation preserving homeomorphisms of the respective domains. The internal points of the simple arc  $\alpha$  are defined by  $\alpha \setminus \{\alpha(0), \alpha(1)\}$  and denoted  $\dot{\alpha}$ . Given distinct points  $B_1, B_2, \dots$ , in  $\mathbb{R}^2$ , we denote by  $B_1 B_2 \dots$  the polygonal arc connecting them by straight segments of lines following that order. We may also denote by  $AB$  a simple arc with endpoints  $A \neq B$ , which is not necessarily a line segment.

- (b) Given any two simple arcs  $\eta_0$  and  $\eta_1$  with a unique common end point, we denote by  $\eta_0 \cup \eta_1$  the simple arc obtained by concatenating  $\eta_0$  and  $\eta_1$  in the usual way and respecting the orientation from  $\eta_0$  to  $\eta_1$ .
- (c) We say that the simple arc  $\alpha \subset \mathbb{R}^2$  is a translation arc if  $\alpha(0) = z$ ,  $\alpha(1) = h(z) \neq z$  and

$$\alpha \cap h(\alpha) = \{h(z)\}.$$

- (d) Let  $\alpha$  be a simple arc with end points  $b$  and  $c$ . We say that  $\alpha$  abuts on its inverse or direct image, respectively, if  $b \notin h^{-1}(\alpha) \cup h(\alpha) = \emptyset$  and one of the following conditions holds:
- (i)  $\dot{\alpha} \cap h^{-1}(\alpha) = \emptyset$  and  $c \in h^{-1}(\alpha)$ .
  - (ii)  $\dot{\alpha} \cap h(\alpha) = \emptyset$  and  $c \in h(\alpha)$ .
- (e) We say that  $L \subset \mathbb{R}^2$  is a Brouwer line for  $h$  if  $L$  is the image of a proper topological embedding  $\psi : \mathbb{R} \rightarrow \mathbb{R}^2$  so that  $h(L)$  and  $h^{-1}(L)$  lie in different components of  $\mathbb{R}^2 \setminus L$ .

Let  $AB$  be a translation arc with end points  $A$  and  $B := h(A)$ . Let  $C = h(B)$  and denote by  $BC$  the simple arc given by  $h(AB)$ . Let us assume without loss of generality that the vertical line passing through  $B$  intersects the arcs  $AB$  and  $BC$  only at  $B$ . Otherwise, we can perform a topological change of coordinates in order to achieve this property.

We will construct two half lines  $L_1$  and  $L_2$  issuing from  $B$ , with  $L_1$  starting upwards and  $L_2$  starting downwards, so that  $L = L_1 \cup L_2$  is a Brouwer line for  $h$ .  $L_1$  and  $L_2$  will be referred to as half Brouwer lines since both are topological embeddings of  $[0, \infty)$  into  $\mathbb{R}^2$  and  $h(L_i) \cap L_i = \emptyset, i = 1, 2$ .

Let us start with  $L_1$ . Consider the vertical arc  $\gamma_1$  starting upwards from  $B$  which is defined by  $\gamma_1(t) = B + (0, t)$ , where  $t \in [0, t^*]$  ( $t^*$  to be defined below), or  $t \in [0, \infty)$ . One of the following conditions is met:

- (i) There exists  $t^* > 0$  such that  $\gamma_1$  abuts on its inverse image and  $P := \gamma_1(t^*)$  is such that  $h(P) =: P'$  is an internal point of  $\gamma_1$ .
- (ii) There exists  $t^* > 0$  such that  $\gamma_1$  abuts on its image and  $P := \gamma_1(t^*)$  is an internal point of  $h(\gamma_1)$ . In this case we set  $P' := h^{-1}(P)$  which is an internal point of  $\gamma_1$ .
- (iii)  $\gamma_1$  is defined for all  $t \geq 0$ ,  $(h^{-1}(\dot{\gamma}_1) \cup \dot{\gamma}_1 \cup h(\dot{\gamma}_1)) \cap (AB \cup BC) = \emptyset$  and  $h(\gamma_1) \cap \gamma_1 = \emptyset$ .

In case (iii) our construction of  $L_1$  ends and we define  $L_1 = \gamma_1$ . Otherwise in cases (i) and (ii), we define  $PP'$  to be the simple arc in  $\gamma_1$  from  $P'$  to  $P$ . Notice that by construction  $PP'$  is a translation arc. Kerékjártó proves the following theorem.

**Theorem 2.2** (See [3], Theorems II, III and IV). *In cases (i) and (ii) above, we have*

$$h(\gamma_1) \cap AB = h^{-1}(\gamma_1) \cap BC = h(\gamma_1) \cap h^{-1}(\gamma_1) = \emptyset.$$

Moreover, in case (i) there exists a sub-arc  $\nu_1$  of  $h^{-1}(\gamma_1)$  from  $A$  to  $P$  such that  $\nu_1 \cup PP' \cup h(\gamma_1) \cup BC \cup AB$  is a simple closed curve which bounds an open domain  $U_1 \subset \mathbb{R}^2$ . In case (ii) there exists a sub-arc  $\nu_1$  of  $h(\gamma_1)$  from  $C$  to  $P$  such that  $\nu_1 \cup PP' \cup h^{-1}(\gamma_1) \cup AB \cup BC$  is a simple closed curve which bounds an open domain  $U_1 \subset \mathbb{R}^2$ .

**Definition 2.3.** The free side of  $PP'$  is defined to be the side of  $PP'$  towards outside  $U_1$  as in Theorem 2.2. See Figure 1.

The free side of the translation arc  $PP' \subset \gamma_1$  only depends on which side  $\gamma_1$  lies with respect to the oriented arc  $AB \cup BC$  and on how  $\gamma_1$  abuts its image according to cases (i) or (ii). This dependence strongly follows from the assumption that  $h$  has no fixed points and is exemplified in Figure 1.

Now we need a couple of definitions in order to start the construction of  $L_1$ .

**Definition 2.4.** (1) Let  $R_n = [-n, n] \times [-n, n]$ ,  $\forall n \in \mathbb{N}^*$  and

$$\epsilon_n = \inf\{|h(x) - x|, |h^{-1}(x) - x| : x \in R_n\} > 0.$$

Define  $\eta_n > 0$  to be the largest number  $t \in (0, \epsilon_n/2]$  so that  $|h(x) - h(y)| \leq \epsilon_n/2$  and  $|h^{-1}(x) - h^{-1}(y)| \leq \epsilon_n/2$ , whenever  $x, y \in R_n$  and  $|x - y| \leq t$ .

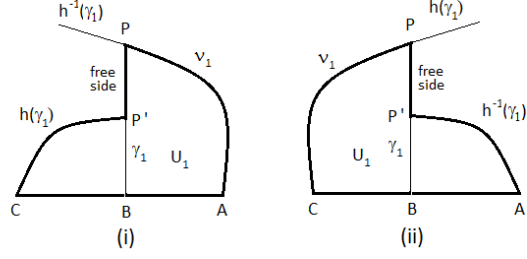


FIGURE 1. In this picture,  $\gamma_1$  abuts on its inverse and direct image as in cases (i) and (ii), respectively.

- (2) Let  $n \in \mathbb{N}^*$  and assume  $PP' \subset R_n$ . By mid-segment of  $PP'$  we mean a segment  $M \subset PP'$  so that the distances of its points to  $P$  and to  $P'$  are at least  $\eta_n$ . Notice that  $M \neq \emptyset$ .
- (3) A base-point associated to the vertical translation arc  $PP'$  and to a given free side of  $PP'$  is a point  $B_1$  in a mid-segment  $M \subset PP'$  such that either the half line  $l_{B_1}$  starting from  $B$  towards the free side of  $PP'$  is such that  $l_{B_1} \cap (h(l_{B_1}) \cup h(PP') \cup h^{-1}(PP')) = \emptyset$  or there exists a simple arc  $\beta$  starting from  $B_1$ , perpendicular to  $PP'$  and towards the free side of  $PP'$  such that  $\beta$  abuts on its image and  $\beta \cap (h(PP') \cup h^{-1}(PP')) = \emptyset$ . In the former case, we say that the base point  $B_1$  with that given free side is unbounded and in the latter case we say that the base point  $B_1$  with that given free side is bounded. One of the endpoints of  $\beta$  is  $B_1$  and the other is denoted by  $P_1$ .

The proof of the existence of at least one base point associated to a translation arc  $PP'$  and to any given free side of  $PP'$  is found in [3, Section 2.2].

**Remark 2.5.** If the translation arc  $PP'$  is horizontal, then the definitions above are the same and analogous results hold.

Continuing our construction, we find a base point  $B_1$  associated to the vertical translation arc  $PP'$ . The initial part of  $L_1$  is then defined to be the segment  $BB_1$ . If  $B_1$  is unbounded then we are finished and  $L_1 = BB_1 \cup l_{B_1}$  is the desired half line. If  $B_1$  is bounded then the horizontal segment  $\beta = B_1P_1$  abuts on its image and we find an internal point  $P'_1 = h(P_1)$  or  $P'_1 = h^{-1}(P_1)$  as before such that the horizontal arc  $P_1P'_1 \subset \beta$  is a translation arc. The translation arc  $P_1P'_1$  admits a free side according to the description above. Observe that now the free side of  $P_1P'_1$  is either the upper or the lower side. Again we find a base point  $B_2 \subset P_1P'_1$  towards the free side of  $P_1P'_1$  and

add the simple arc  $B_1B_2$  to  $L_1$ , now given by  $L_1 = BB_1B_2$ . Repeating this procedure indefinitely we arrive at one of the following cases:

- (i) after a finite number of steps we find an unbounded base point  $B_j \in P_{j-1}P'_{j-1}$  and our broken half line is given by  $L_1 = BB_1B_2 \dots B_j l_{B_j}$ .
- (ii) all base points  $B_j$  found in the construction are bounded and we define  $L_1 = BB_1B_2B_3 \dots$ . Then the following holds: given  $n \in N^*$ , there exists  $k_0 \in N^*$  such that  $B_k \notin R_n, \forall k \geq k_0$ . This follows from the definition of base points and is proved in [3].

Notice that the construction of  $L$  depends on the choices of the internal base points  $B_k \in P_{k-1}P'_{k-1}$ . Also, the half line  $L_1$  goes to infinity and

$$(6) \quad h(L_1) \cap L_1 = (h(\dot{L}_1) \cup h^{-1}(\dot{L}_1) \cup \dot{L}_1) \cap (AB \cup BC) = \emptyset.$$

We still need a modification trick from [3] in the construction of  $L_1$ . It is called the deviation of the path. Let  $V_k = \{(x, y) \in \mathbb{R}^2 : x = k\}, k \in \mathbb{Z}$ , be the vertical lines at integer values and assume that

$$(7) \quad 0 < l := \#V_0 \cap h^{-1}(V_0) < \infty.$$

Notice that from (5), hypothesis (7) must hold as well for each  $V_k, k \in \mathbb{Z}$ , and the respective intersections are shifted by  $(k, 0)$ .

Let  $V_0 \cap h^{-1}(V_0) = \{w_1, \dots, w_l\}$  and  $w'_i := h(w_i), i = 1, \dots, l$ . Consider the vertical arcs  $\gamma_i = w_i w'_i \subset V_0, i = 1, \dots, l$ . If  $\gamma_j$  does not properly contains any  $\gamma_i$  with  $i \neq j$ , then  $\gamma_j$  is a translation arc. We consider only such translation arcs on  $V_0$  and keep denoting them by  $\gamma_j$ , now with  $j = 1, \dots, l_0, l_0 \leq l$ . Given  $j$ , assume a free side of  $\gamma_j$  is given and is to the left. Then there exists a base point  $u_{j,l} \in \dot{\gamma}_j$  associated to  $\gamma_j$  and to that free side. Accordingly, if the given free side of  $\gamma_j$  is to the right we can also find a base point  $u_{j,r} \in \dot{\gamma}_j$  associated to  $\gamma_j$  and to that free side. Let  $\gamma_j^i = \gamma_j + (i, 0)$  be the respective translation arcs on  $V_i$  for all  $i \in \mathbb{Z}$  and let  $u_{j,l}^i := u_{j,l} + (i, 0), u_{j,r}^i := u_{j,r} + (i, 0), i \in \mathbb{Z}$  be their respective base points. In the following we fix these base points  $u_{j,l}^i$  and  $u_{j,r}^i$  in each  $\gamma_j^i$ .

In the construction of  $L_1$  above suppose that at some point we find a vertical translation arc  $P_{k-1}P'_{k-1}$  with a given free side and the horizontal arc issuing from a bounded base point  $B_k \in P_{k-1}P'_{k-1}$  towards the free side intersects some  $V_j$  at an internal point  $z \in B_k B_{k+1}$  so that the arc  $B_k z$  intersects no other vertical  $V_i, i \neq j$ . Instead of adding the segment  $B_k B_{k+1}$  to  $L_1$  we add only the segment  $B_k z$  and the new  $B_{k+1}$  is determined according to one of the alternatives found in the following theorem.

**Theorem 2.6** ([3],[2]). *We have*

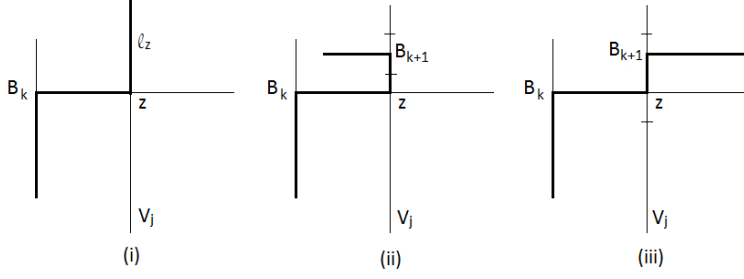


FIGURE 2. The deviation of the path according to cases (i), (ii) and (iii) above.

- (i) *there exists a vertical half line  $l_z \subset V_j$  through  $z$  so that the broken half-line  $\alpha := B_k z \cup l_z$  satisfies  $\alpha \cap h(\alpha) = \emptyset$  and  $(h(P_{k-1}P'_{k-1}) \cup h^{-1}(P_{k-1}P'_{k-1})) \cap \alpha = \emptyset$ . In this case we have  $L_1 = BB_1 \dots B_k z l_z$  and the construction of  $L_1$  is finished.*
- (ii) *there exists  $c \in V_j, c \neq z$ , so that the broken arc  $\alpha := B_k z \cup z c$  abuts on its image, satisfies  $(h(P_{k-1}P'_{k-1}) \cup h^{-1}(P_{k-1}P'_{k-1})) \cap \alpha = \emptyset$ , and contains a translation arc  $\gamma_m^j \subset V_j$  for some  $m \in \{1, \dots, l_0\}$ . Let  $B_{k+1} \in \{u_{m,l}^j, u_{m,r}^j\}$  be the base point in  $\gamma_m^j$  associated to the free side of  $\alpha$ . In this case we have  $L_1 = BB_1 \dots B_k z B_{k+1} \dots$  and we keep constructing  $L_1$  through the horizontal arc issuing from  $B_{k+1} \in V_j$  towards the free side of  $\alpha$  as before.*
- (iii) *the point  $z$  is an internal point of a translation arc  $\gamma_m^j$  for some  $m$  so that  $\gamma_m^j \cap (h(P_{k-1}P'_{k-1}) \cup P_{k-1}P'_{k-1} \cup h^{-1}(P_{k-1}P'_{k-1})) = \emptyset$  and  $\cup_{n \in \mathbb{Z}} h^n(\gamma_m^j) \cap B_k z = \{z\}$ . In this case the free side of  $\gamma_m^j$  is the side opposite to  $B_k z$  and we find a base point  $B_{k+1} \in \{u_{m,l}^j, u_{m,r}^j\}$  associated to  $\gamma_m^j$  and to that free side. We have  $L_1 = BB_1 \dots B_k z B_{k+1} \dots$  and we keep constructing  $L_1$  through the horizontal arc issuing from  $B_{k+1}$  towards that free side.*

Proceeding as above indefinitely and using the deviation of the path whenever its conditions are met we find the desired half line  $L_1$ . As explained before,  $L_1$  goes to infinity and satisfies (6).

**Remark 2.7.** Given  $a \in \mathbb{R}$ , let  $G_a = \{(x, y) \in \mathbb{R}^2 : y \geq a\}$ . We can extract some more information on how  $L_1$  goes to infinity if we assume the following twist condition on  $h$

$$(8) \quad \exists y_0 \in \mathbb{R} \text{ s.t. } p_1 \circ h(x, y) > x \text{ and } p_1 \circ h^{-1}(x, y) < x, \forall (x, y) \in G_{y_0}.$$

Suppose that at some point in the construction of  $L_1$  we find a horizontal translation arc  $P_{k-1}P'_{k-1}$  with the free side coinciding to its upside and an



associated base point  $B_k \in P_{k-1}P'_{k-1}$ . Assume that there exists a vertical segment  $V$  starting from  $B_k$  towards the free side so that its other extremity lies inside  $G_{y_0}$  and that  $h(V) \cap V = \emptyset$ . We claim that  $B_k$  is an unbounded base point and the construction of  $L_1$  ends by adding to it the vertical half line  $l_{B_k}$ , i.e.,  $L_1 = BB_1 \dots B_k l_{B_k}$ . To see this we argue indirectly and assume the existence of a vertical segment  $W$  starting from  $B_k$  and containing  $V$  such that  $W$  abuts on its image. Let  $w \neq B_k$  be the other extremity of  $W$ . Then either  $h(w) \in W$  or  $h^{-1}(w) \in W$ . However, this contradicts (8) and proves our claim.

**Remark 2.8.** Using the deviation of the path explained above we know that if  $L_1$  is horizontally unbounded then  $L_1$  is eventually periodic. This follows from the finiteness of the points  $u_{j,l}^i, u_{j,r}^i \in V_i$  for each  $i \in \mathbb{Z}$ . For instance, suppose that  $L_1$  is deviated at some  $V_i$ , for some  $i \in \mathbb{Z}$ , and leaves it to the right at  $u_{j,r}^i \in V_i$ , for some  $j \in \{1, \dots, l_0\}$ . Suppose that after this deviation,  $L_1$  is now deviated at  $V_{i+N}$ , for some  $N \in \mathbb{Z}^*$ , leaving it to the right at  $u_{j,r}^{i+N} \in V_{i+N}$ . We continue the construction of  $L_1$  from  $u_{j,r}^{i+N}$ , proceeding in exactly the same way as we did from  $u_{j,r}^i$ . This implies that, except perhaps for its initial segments,  $L_1$  is periodic, i.e., there exists a connected subset  $W_0 \subset L_1$  from  $u_{j,r}^i$  to  $u_{j,r}^{i+N}$  so that  $W_0 + (kN, 0) \subset L, \forall k \in \mathbb{N}$ . Hence we find a new periodic Brouwer line  $L_{\text{per}}$  given by  $W_0 + (kN, 0), k \in \mathbb{Z}$ . By construction we must have  $h(L_{\text{per}}) \cap L_{\text{per}} = \emptyset$ .

**Remark 2.9.** The construction of the other half line  $L_2$  from  $B$  (now starting downwards) can be done in exactly the same way as we did for  $L_1$  so that by construction  $L = L_1 \cup L_2$  is a Brouwer line. An alternative construction for  $L_2$ , which will be used in the proof of Theorem 1.2, is the following: let  $\psi : [0, \infty) \rightarrow \mathbb{R}^2$  be a proper topological embedding with  $\psi(0) = B$  and let  $L_2 = \psi([0, \infty))$ . Assume the  $h(L_2) \cap L_2 = \emptyset$  and that  $\dot{L}_1$  and  $\dot{L}_2$  lie in different components of  $\mathbb{R}^2 \setminus (h^{-1}(L_2) \cup AB \cup BC \cup h(L_2))$ . Then one easily checks that  $L = L_1 \cup L_2$  is a Brouwer line for  $h$ .

We end this section with a proposition that will be useful in the proof of Theorem 1.2 in the next section. Its proof is entirely contained in Kerékjártó's construction of Brouwer lines explained above.

**Proposition 2.10.** *Let  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an orientation preserving homeomorphism of the plane which has no fixed points and satisfies the following assumptions:*

- (i) *There exists  $y_0 \in \mathbb{R}$  such that*

$$p_1 \circ h(x, y) > x \text{ and } p_1 \circ h^{-1}(x, y) < x, \forall (x, y) \in \mathbb{R}^2, y \geq y_0.$$

- (ii)  $h(x+1, y) = h(x, y) + (1, 0), \forall (x, y) \in \mathbb{R}^2$ .
- (iii) *There exists a vertical line  $V_0$  such that  $0 < l := \#V_0 \cap h^{-1}(V_0) < \infty$ .*

*Then through any point  $B \in \mathbb{R}^2$  as above, there exists a half Brouwer line  $L_1$  issuing from  $B$  upwards so that the following holds:*

- $L_1$  contains only horizontal and vertical segments.
- if  $L_1$  contains a point  $q \in \{y \geq y_0\}$  then it contains the vertical upper half line through  $q$ .
- if  $L_1$  is horizontally unbounded then  $L_1$  is eventually periodic, i.e., there exists a simple arc  $W_0 \subset L_1$  and an integer  $N \neq 0$  so that  $W_0 + (kN, 0) \subset L_1, \forall k \in \mathbb{N}$ .  $|N|$  is the least positive integer with this property. In particular, this implies that  $W_0 \cap W_0 + (N, 0) = \{\text{point}\}$ .
- if  $L_2$  is a given half Brouwer line issuing from  $B$  downwards and  $\dot{L}_1$  and  $\dot{L}_2$  lie in different components of  $\mathbb{R}^2 \setminus (h^{-1}(L_2) \cup AB \cup BC \cup h(L_2))$ , then  $L_1 \cup L_2$  is a Brouwer line.

Here, as above,  $B = h(A), C = h(B)$ ,  $AB$  is a translation arc and  $BC = h(AB)$  is horizontal.

### 3. PROOF OF THEOREM 1.2

We start with the following lemma.

**Lemma 3.1.** *Let  $\tilde{g} : \tilde{U} \subset \tilde{A} \rightarrow \tilde{A}$  be a real analytic area-preserving diffeomorphism defined in an open neighbourhood  $\tilde{U}$  of  $\mathbb{R} \times \{0\} \subset \tilde{A}$  so that  $\text{Fix}(\tilde{g}) = \mathbb{R} \times \{0\}$ . Assume that there exists a sequence of positive real numbers  $\epsilon_n \rightarrow 0^+$  such that each  $\tilde{g}_{\epsilon_n}$ , defined as in (2), admits a fixed point  $p_n \in \tilde{U}, \forall n$ , with  $p_n \rightarrow \bar{p} = (\bar{x}, 0) \in \mathbb{R} \times \{0\}$  as  $n \rightarrow \infty$ . Then there exists a real-analytic curve  $\gamma_0 : [0, 1] \rightarrow \tilde{U}$ ,  $\gamma_0(t) = (x(t), y(t))$  so that  $\tilde{g} \circ \gamma_0(t) = (w(t), y(t))$  and it satisfies*

$$(9) \quad w(0) = \bar{x}, w(t) < x(t) \text{ and } y'(t) > 0, \forall t \in (0, 1].$$

*Proof.* Let us write  $\tilde{g}(x, y) = (g_1(x, y), g_2(x, y))$  and let  $G_2(x, y) := g_2(x, y) - y$ . We may express  $G_2$  as a power series in  $x - \bar{x}$  and  $y$  near  $\bar{p}$  which converges in  $B_\epsilon := \{(x, y) \in \mathbb{R}^2 : (x - \bar{x})^2 + y^2 \leq \epsilon^2\}$  with  $\epsilon > 0$  small.

If  $G_2$  vanishes identically then  $g_2(x, y) = y$  near  $\bar{p}$ . By preservation of area and the fact that  $g_1(x, 0) = (x, 0), \forall x$ , we have  $g_1(x, y) = x + yR(y)$  for a real-analytic function  $R$  defined near  $y = 0$ . Since  $\text{Fix}(\tilde{g}) = \mathbb{R} \times \{0\}$ ,  $R$  does not vanish identically. The existence of  $p_n$  as in the hypothesis implies  $R(y) < 0$  for  $y$  small. In this case we can define the curve  $\gamma_0$  by  $\gamma_0(t) = (\bar{x}, t)$ , with  $t \geq 0$  small.

Now assume that  $G_2$  does not vanish identically. We investigate the zeros of  $G_2$  near  $\bar{p}$  in  $B_\epsilon$  for  $\epsilon > 0$  small. Notice that  $\bar{p} \in \mathbb{R} \times \{0\} \cap B_\epsilon \subset \{G_2 = 0\}$  and thus  $\bar{p}$  is not an isolated point of  $\{G_2 = 0\}$ . Since  $G_2$  is real-analytic we take  $\epsilon > 0$  small and find an even number of real-analytic embedded curves  $\eta_i : [0, 1] \rightarrow B_\epsilon$ ,  $i = 1 \dots 2m$ , with  $\eta_i(0) = \bar{p}$ , so that  $\{G_2 = 0\} \cap B_\epsilon = \cup_{i=1}^{2m} \text{Image}(\eta_i)$ , see lemmas 3.1 and 3.3 of [4]. Taking  $\epsilon > 0$  even smaller, we may assume that the image of any two of these curves intersect each other only at  $\bar{p}$ . Also, we may choose  $\eta_1(t) = (\bar{x} + \epsilon t, 0)$  and  $\eta_2(t) = (\bar{x} - \epsilon t, 0)$ ,  $t \in [0, 1]$ , since  $\mathbb{R} \times \{0\} \subset \{G_2 = 0\}$ . The existence of the sequence  $p_n \rightarrow \bar{p}$  as in the hypothesis implies that  $m \geq 2$  and therefore we find  $j_0 \in \{3, \dots, 2m\}$  and a subsequence of  $p_n$ , still denoted by  $p_n$ , such that  $p_n \in \text{Image}(\eta_{j_0})$ . Moreover, since  $\eta_{j_0}(t) = (x_{j_0}(t), y_{j_0}(t))$  is real analytic, we have  $y'_{j_0}(t) > 0$ ,  $\forall t \in (0, \mu]$ , for some  $\mu > 0$  small, and, therefore,  $\text{Image}(\eta_{j_0}|_{[0, \mu]})$  projects injectively into the  $y$ -axis. Finally, we define  $\gamma_0(t) = \eta_{j_0}(\mu t)$ ,  $t \in [0, 1]$ . By the properties of  $p_n$  and the fact that  $\text{Fix}(\tilde{g}) = \mathbb{R} \times \{0\}$ , we get that  $\gamma_0$  satisfies the desired properties as in the statement.  $\square$

To prove Theorem 1.2 we argue indirectly. Assume that there exists a sequence  $\epsilon_n \rightarrow 0^+$  so that  $\tilde{f}_{\epsilon_n}$ , defined as in (2), admits a fixed point  $p_n$ . By the periodicity of  $p_1 \circ \tilde{f}(x, y) - x$  in  $x$  we can assume that  $p_n \rightarrow \bar{p} = (\bar{x}, 0) \in \mathbb{R} \times \{0\}$ . This implies that  $\tilde{f}$ , restricted to a neighbourhood  $\tilde{U}$  of  $\mathbb{R} \times \{0\}$ , satisfies the conditions of Lemma 3.1. So we find a real-analytic curve  $\gamma_0 : [0, 1] \rightarrow \tilde{A}$ ,  $\gamma_0(t) = (x(t), y(t))$ , so that  $\tilde{f} \circ \gamma_0(t) = (w(t), y(t))$  satisfies (9). In what follows, the curve  $\gamma_0$  will be prolonged to a Brouwer line  $\tilde{L}$  in  $\tilde{A}$  satisfying one of the possibilities:

- $\tilde{L}$  hits  $\mathbb{R} \times \{1\}$ . Since  $\tilde{f}$  moves points in  $\mathbb{R} \times \{1\}$  to the right and moves  $\gamma_0$  to the left,  $\tilde{L}$  must intersect its image, a contradiction.
- $\tilde{L}$  is eventually periodic. In this case we obtain an area-preserving diffeomorphism of the closed annulus with a homotopically non-trivial simple closed curve which is disjoint from its image, again a contradiction.
- $\tilde{L}$  is bounded and accumulates at  $\mathbb{R} \times \{0\}$ . In this case we obtain an area-preserving homeomorphism of the 2-sphere admitting a simple closed curve bounding a topological disk whose image is properly contained inside itself, a contradiction.

Given  $t \in (0, 1]$  let  $B_t := \gamma_0(t)$ ,  $C_t := \tilde{f}(B_t)$  and  $A_t := \tilde{f}^{-1}(B_t)$ . Let  $B_t C_t$  be the horizontal segment connecting  $B_t$  and  $C_t$ , and let  $A_t B_t := \tilde{f}^{-1}(B_t C_t)$  be its inverse image. We claim that  $A_t B_t$  is a translation arc for  $\tilde{f}$  if  $t$  is small enough. To see this, assume this is not the case so that

we can find a point  $B_t \neq z \in B_t C_t$  which is also contained in  $A_t B_t$ . It follows that  $p_1(z) > p_1(B_t)$ . Since  $\frac{\partial(p_1 \circ \tilde{f})}{\partial x}(x, y) \rightarrow 1$  as  $(x, y) \rightarrow \bar{p}$ , we have  $p_1 \circ \tilde{f}(z) - p_1 \circ \tilde{f}(B_t) = \frac{\partial(p_1 \circ \tilde{f})}{\partial x}(\xi)(p_1(z) - p_1(B_t)) > 0$ , for some  $\xi \in B_t C_t$  and  $t > 0$  small. Since  $\tilde{f}(z) \in B_t C_t$ , we also have  $p_1 \circ \tilde{f}(z) \leq p_1(C_t) = p_1 \circ \tilde{f}(B_t)$ . This leads to a contradiction which proves that indeed  $A_t B_t$  is a translation arc for  $\tilde{f}$  and

$$(10) \quad p_1(z) > p_1(B_t),$$

for all internal points  $z \in A_t B_t$  where  $t \in (0, 1]$  is fixed and small.

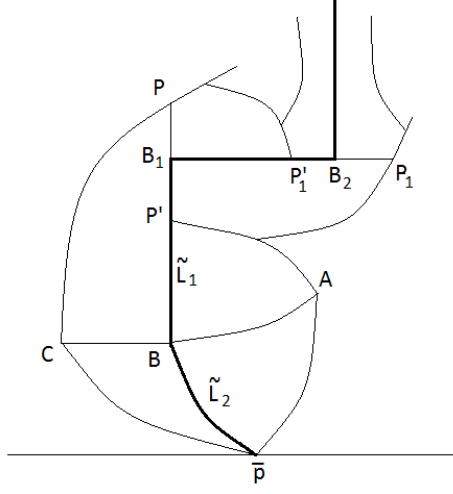
Let us fix a sufficiently small  $t_0 > 0$  such that for some number  $c_0 \in \mathbb{R}$ ,  $\tilde{f}^{-1}(\gamma_0([0, t_0])) \cup \gamma_0([0, t_0]) \cup \tilde{f}(\gamma_0([0, t_0])) \cup B_{t_0} C_{t_0} \cup A_{t_0} B_{t_0}$  is disjoint from all the verticals  $V_{k+c_0} = \{(x, y) \in \tilde{A} : x = k + c_0\}, k \in \mathbb{Z}$ . We may assume without loss of generality that  $c_0 = 0$ .

In order to directly apply results from Kerékjártó's construction of Brouwer lines in the plane as stated in Proposition 2.10, we consider the homeomorphism  $d : \text{interior}(\tilde{A}) \rightarrow \mathbb{R}^2$  given by  $d(x, y) = (x, \frac{y-1/2}{y(1-y)})$  and the induced orientation preserving homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $h = d \circ \tilde{f} \circ d^{-1}$ . From the hypothesis  $\text{Fix}(\tilde{f}) = \mathbb{R} \times \{0\}$ , we get  $\text{Fix}(h) = \emptyset$ .

Let  $A := d(A_{t_0})$ ,  $B := d(B_{t_0})$  and  $C := d(C_{t_0})$ . Denote by  $AB$  the simple arc  $d(A_{t_0} B_{t_0})$  and by  $BC$  its image under  $h$ . Notice that  $AB$  is a translation arc and that  $BC$  is a horizontal simple arc. From (10), the vertical line through  $B$  intersects  $AB$  and  $BC$  only at  $B$ . Hence we can start the construction of a Brouwer line for  $h$  with the vertical line starting from  $B$  towards the upside and proceeding as in Section 2, thus obtaining the half line  $L_1$ . To obtain  $L_2$  we simply define it by  $L_2 = d(\gamma_0|_{(0, t_0]})$ . It follows that  $L = L_1 \cup L_2$  is a Brouwer line, see Remark 2.9. Let  $\tilde{L} := d^{-1}(L) = \tilde{L}_1 \cup \tilde{L}_2$  and observe that

$$(11) \quad \tilde{f}(\tilde{L}) \cap \tilde{L} = \emptyset$$

Now we prove that the existence of the Brouwer line  $\tilde{L}$  leads to a contradiction. First, from the twist condition (1), we can find  $0 < \delta < 1$  such that for all  $(x, y) \in S_\delta := \{(x, y) \in \tilde{A} : \delta \leq y \leq 1\}$ , we have  $p_1 \circ \tilde{f}(x, y) > x$  and  $p_1 \circ \tilde{f}^{-1}(x, y) < x$ . This implies that  $h$  satisfies condition (8) for  $y_0 = \frac{\delta-1/2}{\delta(1-\delta)}$ , see Remark 2.7. It follows that if  $\tilde{L}$  hits  $S_\delta$  then  $\tilde{L}$  contains a vertical segment with end point  $z_0 \in \mathbb{R} \times \{1\}$ . By construction, points of  $\tilde{L}$  near but different from  $\bar{p}$  are mapped under  $\tilde{f}$  to the left, while points near  $z_0$  are mapped to the right. This implies that  $\tilde{f}(\tilde{L}) \cap \tilde{L} \neq \emptyset$ , which contradicts (11). Hence we can assume that  $\tilde{L}$  does not accumulate at  $\mathbb{R} \times \{1\}$ .


 FIGURE 3. The Brower line  $\tilde{L} = \tilde{L}_1 \cup \tilde{L}_2 \subset \tilde{A}$ .

Since  $\tilde{f}$  is analytic and  $p_1 \circ \tilde{f}(x, 1) > x, \forall x \in \mathbb{R}$ , we get that  $\tilde{f}^{-1}(V_k) \cap V_k = (\tilde{f}^{-1}(V_0) \cap V_0) + (k, 0)$  is a finite set for all  $k \in \mathbb{Z}$  and, therefore, condition (7) holds for  $h$ . This implies that if  $\tilde{L}$  is horizontally unbounded then, as explained in Remark 2.8, we can find  $N \in \mathbb{Z}$  and another Brouwer line  $\tilde{L}_{\text{per}} = d^{-1}(L_{\text{per}}) \subset \text{interior}(\tilde{A})$  which is  $N$ -periodic in  $x$ , i.e.,  $\tilde{L}_{\text{per}} + (kN, 0) = \tilde{L}_{\text{per}}, \forall k \in \mathbb{Z}$ . Let  $f_N : A_N \rightarrow A_N$  be the map induced by  $\tilde{f}$  on the annulus  $A_N := \tilde{A}/T_N$ , where  $T_N : \tilde{A} \rightarrow \tilde{A}$  is the horizontal translation  $T_N(x, y) = (x + N, y)$ . Let  $p_N : \tilde{A} \rightarrow A_N$  be the associated covering map and let  $L_N := p_N(\tilde{L}_{\text{per}})$ . From the properties of  $\tilde{L}_{\text{per}}$  and of the map  $\tilde{f}$  we see that  $L_N$  and  $f_N(L_N)$  are disjoint simple closed curves which are homotopically non-trivial. Let  $C_-$  be the topological closed annulus bounded by  $L_N$  and  $p_N(\mathbb{R} \times \{0\})$ . Then either  $f_N$  or  $f_N^{-1}$  maps  $C_-$  properly into itself. Since  $f_N$  preserves a finite area form, we get a contradiction. Hence we can assume that  $\tilde{L}$  is horizontally bounded and accumulates only at  $\mathbb{R} \times \{0\}$ .

Now we find  $N_0 \in \mathbb{N}$  large enough so that  $\tilde{L} \cap (\tilde{L} + (N_0, 0)) = \emptyset$ , which implies by Brouwer's lemma (see for instance [1]) that

$$(12) \quad \tilde{L} \cap (\tilde{L} + (kN_0, 0)) = \emptyset, \forall k \in \mathbb{Z}^*.$$

As before we consider the annulus  $A_{N_0} := \tilde{A}/T_{N_0}$  and identify the points in each component of  $\partial A_{N_0}$  to obtain a topological sphere  $S^2$ . We end up with a map  $\widehat{f}_{N_0} : S^2 \rightarrow S^2$  induced by  $f_{N_0}$  which preserves orientation and a finite area form. The closure of  $p_{N_0}(\tilde{L})$  corresponds to a simple closed curve  $\gamma_0 \subset S^2$  passing through the pole  $p_0$ , which corresponds to the lower component of  $\partial A_{N_0}$ . This last assertion follows from (12). Since  $\tilde{L}$  is a Brouwer line we see that  $\widehat{f}_{N_0}(\gamma_0) \cap \gamma_0 = \{p_0\}$  and that  $\widehat{f}_{N_0}$  maps properly

one component of  $S^2 \setminus \gamma_0$  into itself. This contradicts the preservation of a finite area form and shows that  $\tilde{L}$  cannot exist. The proof of Theorem 1.2 is complete.

#### 4. PROOF OF THEOREM 1.5

Our aim in this section is to construct an annulus diffeomorphism  $f : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$ , homotopic to the identity map, which satisfies

- (i)  $f$  is smooth and area-preserving.
- (ii)  $\text{Fix}(f) = S^1 \times \{0\}$ .
- (iii) If  $\tilde{f} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$  is the lift of  $f$  satisfying  $\tilde{f}(x, 0) = (x, 0), \forall x \in \mathbb{R}$ , then  $p_1 \circ \tilde{f}(x, 1) > x, \forall x \in \mathbb{R}$ .
- (iv) Given  $\epsilon > 0$ , if  $f_\epsilon : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$  is the map induced by the lift  $\tilde{f}_\epsilon := \tilde{f} + (\epsilon, 0)$  as before, then there exists a positive sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  with  $\epsilon_n \rightarrow 0^+$  as  $n \rightarrow \infty$ , such that  $\text{Fix}(\tilde{f}_{\epsilon_n}) \neq \emptyset, \forall n$ .

As proved in Theorem 1.2, such a diffeomorphism cannot exist if smoothness is replaced by real-analyticity in (i).

**4.1. Area-preserving maps and generating functions.** We start by recalling basic facts on area preserving maps associated to generating functions. Let  $U := \{(X, y) \in \mathbb{R} \times [0, 1] : X^2 + y^2 < \varepsilon\}$ ,  $\varepsilon > 0$ , and  $g : U \rightarrow \mathbb{R}$  be a smooth function so that

$$(13) \quad D^\nu g|_{\{y=0\} \cap U} \equiv 0, \forall 0 \leq |\nu| \leq 2.$$

We denote by  $G : U \rightarrow \mathbb{R}$  the function given by

$$(14) \quad G(X, y) := Xy - g(X, y).$$

Let  $(x, Y) \in \mathbb{R} \times [0, 1]$  be given by

$$(15) \quad \begin{aligned} x &:= G_y = X - g_y(X, y), \\ Y &:= G_X = y - g_X(X, y). \end{aligned}$$

We see from the first equation of (15) and the hypothesis (13) on  $g$  that we can use the implicit function theorem to write  $X = \alpha(x, y)$  for  $|(x, y)|$  small, where  $\alpha$  is a smooth map satisfying  $\alpha(x, 0) = x, \forall x$ . In this case  $Y = y + g_X(\alpha(x, y), y) = \beta(x, y) \geq 0$ , where  $\beta$  is smooth and satisfies  $\beta(x, 0) = 0, \forall x$ . Let us denote by  $\bar{f} : V \subset \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$  the map given by

$$(X, Y) = \bar{f}(x, y) := (\alpha(x, y), \beta(x, y)),$$

where  $V$  is a small neighborhood of  $(0, 0) \in \mathbb{R} \times [0, 1]$ . We say that  $\bar{f}$  is a local map associated to the generating function  $G$ . Moreover,  $\bar{f}|_{\mathbb{R} \times \{0\}}$  is the identity map.

**Proposition 4.1.** *The map  $\bar{f}$  preserves the area form  $dx \wedge dy$  on  $\mathbb{R} \times [0, 1]$ , i.e.,*

$$dX \wedge dY = dx \wedge dy.$$

*Proof.* From (15) we have

$$(16) \quad \begin{aligned} dx &= (1 + g_{yX})dX + g_{yy}dy \Rightarrow dx \wedge dy = (1 + g_{yX})dX \wedge dy, \\ dY &= (1 + g_{Xy})dy + g_{XX}dX \Rightarrow dX \wedge dY = (1 + g_{Xy})dX \wedge dy. \end{aligned}$$

Since  $g$  is smooth, the proposition follows.  $\square$

**4.2. A special generating function.** Let  $\rho : [0, \infty) \rightarrow [0, 1]$  be a smooth function satisfying  $\rho \equiv 1$  in  $[0, \frac{1}{16}]$ ,  $\rho \equiv 0$  in  $[\frac{1}{4}, \infty)$  and  $\rho' < 0$  in  $(\frac{1}{16}, \frac{1}{4})$ . We define the vector field  $X$  on the strip  $W_1 := \mathbb{R} \times [-1, 1]$  by

$$X(x, y) = \rho(x^2 + y^2) \cdot (-y, x).$$

It is clear that  $X$  is smooth,  $X \equiv 0$  outside  $B(1/2) := \{(x, y) \in W_1 : x^2 + y^2 \leq 1/4\}$  and  $X(x, y) = (-y, x)$  in  $B(1/4)$ .

The flow  $\{\varphi_t\}$  of  $X$  on  $W_1$  is defined for all  $t \in \mathbb{R}$  and satisfies

$$(17) \quad \begin{aligned} \varphi_t(x, y) &= (x, y), \forall (x, y) \in W_1 \setminus B(1/2), \forall t, \\ \varphi_\pi(x, y) &= -(x, y), \forall (x, y) \in B(1/4). \end{aligned}$$

Now let  $W_0 := \mathbb{R} \times [0, 1]$ . For each  $k \in \mathbb{N}$ , let  $F_k := \mathbb{R} \times (\frac{1}{2^{k+1}}, \frac{1}{2^k}] \subset W_0$ ,  $F_\infty := \mathbb{R} \times \{0\} \subset W_0$  and consider the diffeomorphism

$$(18) \quad \begin{aligned} t_k : F_k &\rightarrow W_1 \setminus \mathbb{R} \times \{-1\}, k \in \mathbb{N}, \\ (x, y) &\mapsto (2^{k+2}x, 2^{k+2}y - 3). \end{aligned}$$

Letting  $\partial_k^+ := \mathbb{R} \times \{1/2^k\} \subset F_k$ , we observe that  $t_k(\partial_k^+) = \mathbb{R} \times \{1\}$ .

Next let us define a map  $\psi : W_0 \rightarrow W_0$  by

$$(19) \quad \begin{aligned} \psi|_{F_k} &:= t_k^{-1} \circ \varphi_\pi \circ t_k, \forall k, \\ \psi|_{F_\infty} &:= \text{Id}|_{F_\infty}. \end{aligned}$$

Let  $p_k := (0, \frac{3}{2^{k+2}}) \in F_k \subset W_0$  be the ‘midpoint’ of  $F_k$ . From (17), (18) and (19) we note that

$$(20) \quad \text{supp} \psi = \overline{\bigcup_{k \geq 0} B_{p_k}(1/2^{k+3})},$$

where  $B_p(r)$  denotes the closed ball centered at  $p$  with radius  $r$ . Moreover,  $\psi$  is the identity map when restricted to a small neighbourhood of each  $\partial_k^+, \forall k$ . These observations together with the second equation in (19) imply that  $\psi$  is smooth in  $W_0 \setminus \{(0, 0)\}$ . We also get that  $\psi$  is a diffeomorphism when restricted to  $W_0 \setminus \{(0, 0)\}$ . From the second equation in (17) we have

$$(21) \quad \psi(x, y) = 2p_k - (x, y), \forall (x, y) \in B_{p_k}(1/2^{k+4}), \forall k.$$

Let  $h : [0, 1] \rightarrow [0, \infty)$  be the smooth function given by

$$(22) \quad \begin{aligned} h(t) &= e^{-1/t}, \forall t > 0, \\ h(0) &= 0. \end{aligned}$$

Note that  $h$  is flat at  $t = 0$ , i.e.,

$$(23) \quad h^{(n)}(0) = 0, \forall n.$$

Observe also that given  $l \in \mathbb{N}$ , we find polynomial functions  $P_l, Q_l$  such that

$$(24) \quad h^{(l)}(t) = e^{-1/t} \frac{P_l(t)}{Q_l(t)}, \forall t > 0.$$

This can easily be proved by induction.

Now let  $g : W_0 \rightarrow [0, \infty)$  be defined by

$$(25) \quad g := h \circ p_2 \circ \psi.$$

**Proposition 4.2.** *We have the following:*

- (i) *The function  $g$  is smooth and  $D^\nu g|_{\mathbb{R} \times \{0\}} \equiv 0, \forall \nu \geq 0$ .*
- (ii) *The set  $\text{Crit}(g)$  of critical points of  $g$  coincides with  $\mathbb{R} \times \{0\}$ .*
- (iii) *There exists a positive sequence  $(s_k)_{k \in \mathbb{N}}$  satisfying  $s_k \rightarrow 0^+$  as  $k \rightarrow \infty$  such that  $\nabla g|_{\partial_k^+} = (0, s_k), \forall k$ .*
- (iv) *There exists a positive sequence  $(m_k)_{k \in \mathbb{N}}$  satisfying  $m_k \rightarrow 0^+$  as  $k \rightarrow \infty$  such that  $\nabla g(p_k) = (0, -m_k), \forall k$ .*

Postponing its proof to Section 4.3 below, we use Proposition 4.2 to show that  $g$  induces a diffeomorphism  $f : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$  satisfying conditions (i)-(iv) as described in the beginning of Section 4.

Let  $G(X, y) = Xy + g(X, y)$  be the function defined in (14). Then, as explained before, we find a small neighbourhood  $V$  of  $(0, 0)$  in  $\mathbb{R} \times [0, 1]$  and a smooth area-preserving map  $\bar{f} : V \rightarrow \mathbb{R} \times [0, 1]$ ,  $\bar{f}(x, y) = (X, Y)$ , so that  $\bar{f}|_{V \cap F_\infty}$  is the identity map, i.e.,  $\bar{f}$  is the local map associated to the generating function  $G$ . From (15) we see that the fixed points of  $\bar{f}$  correspond to critical points of  $g$ . This implies that  $\text{Fix}(\bar{f}) = \text{Crit}(g) = V \cap F_\infty$ . Now observe that since  $\nabla g|_{\partial_k^+} = (0, s_k)$ , with  $s_k \rightarrow 0^+$  as  $k \rightarrow \infty$ , we have from (15) that  $\bar{f}(x, y) = (x + s_k, y), \forall (x, y) \in \partial_k^+$ . In the same way, since  $\nabla g(p_k) = (0, -m_k)$ , we have  $\bar{f}(m_k, 3/2^{k+2}) = p_k = (0, 3/2^{k+2}), \forall k$ , with  $m_k \rightarrow 0^+$  as  $k \rightarrow \infty$ . This implies that for all  $k \geq 0$ , the map  $\bar{f} + (m_k, 0)$  admits  $(m_k, 3/2^{k+2})$  as a fixed point. From (20) and the definition of  $g$ , we see that given any  $x_1 > 0$  small we have  $\bar{f}(x, y) = (x + h'(y), y), \forall (x, y) \in \{|x| = x_1, 0 \leq y \leq 2x_1\}$ . Given  $\lambda > 0$ , let  $T_\lambda : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R} \times [0, \infty)$  be the map  $T_\lambda = (\lambda x, \lambda y)$ . If necessary we replace  $\bar{f}$  by  $(T_{1/2^{k_0}})^{-1} \circ \bar{f} \circ T_{1/2^{k_0}}$ , for a fixed  $k_0$  sufficiently large, in order to find a map defined in  $[-1/2, 1/2] \times [0, 1]$



with the same properties above. Identifying  $(-1/2, y) \in \{-1/2\} \times [0, 1]$  with  $(1/2, y) \in \{1/2\} \times [0, 1]$  we finally find an annulus map  $f : S^1 \times [0, 1] \rightarrow S^1 \times [0, 1]$  with all the desired properties.

Notice that the diffeomorphism induced by the generating function  $g$  is defined in the open neighbourhood  $V \subset \mathbb{R} \times [0, 1]$  which might be very small. This explains why property (iii) is necessary in Proposition 4.2. Its proof is not straightforward and is left for the next section.

**4.3. Proof of Proposition 4.2.** As observed before,  $\psi$  is smooth on  $W_0 \setminus \{(0, 0)\}$ . Hence  $g$  is smooth in this set as well. Moreover, since  $\psi$  is the identity map near  $(\bar{x}, 0)$ , for each  $\bar{x} \neq 0$ , we have that  $g$  is given by  $g(x, y) = h(y)$  near  $(\bar{x}, 0)$ . It follows from (23) that

$$(26) \quad D^\nu g(\bar{x}, 0) = 0, \forall \bar{x} \neq 0, \forall |\nu| \geq 0.$$

It remains to prove that  $g$  is smooth at  $(0, 0)$  and that  $D^\nu g(0, 0) = 0, \forall |\nu| \geq 0$ . Let  $p = p_2 \circ \psi$ . From the definition of  $\psi$  we see that

$$(27) \quad p(x, y) = \frac{1}{2^n} p(2^n x, 2^n y), \forall (x, y) \in W_0 \setminus F_\infty.$$

For any given smooth function  $a : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ , we denote by  $D^\alpha a = \frac{\partial^{|\alpha|} a}{\partial x^i \partial y^j}$  where  $\alpha = (i, j) \in \mathbb{N}^2$  and  $|\alpha| = i + j$ .

**Lemma 4.3.** *In  $W_0 \setminus F_\infty$ , we have*

$$(28) \quad D^\alpha g = \sum_{l=1}^{|\alpha|} h^{(l)}(p) T_{\alpha, l}(p_x, p_y, \dots, D^\beta p),$$

where  $T_{\alpha, l}$  is a multi-variable polynomial function on  $D^\beta p$  with  $\beta$  satisfying  $1 \leq |\beta| \leq |\alpha| - l + 1$ .

*Proof.* Observe that  $D^{(1,0)}g = h'(p)p_x$  and  $D^{(0,1)}g = h'(p)p_y$  which have the form above with

$$\begin{aligned} T_{(1,0),1}(p_x, p_y) &= p_x \\ T_{(0,1),1}(p_x, p_y) &= p_y. \end{aligned}$$

In the same fashion we have  $D^{(2,0)}g = h'(p)p_{xx} + h''(p)p_x^2$ ,  $D^{(1,1)}g = h'(p)p_{xy} + h''(p)p_x p_y$ ,  $D^{(0,2)}g = h'(p)p_{yy} + h''(p)p_y^2$  and

$$\begin{aligned} T_{(2,0),1}(p_x, p_y, p_{xx}, p_{xy}, p_{yy}) &= p_{xx}, & T_{(2,0),2}(p_x, p_y) &= p_x^2, \\ T_{(1,1),1}(p_x, p_y, p_{xx}, p_{xy}, p_{yy}) &= p_{xy}, & T_{(1,1),2}(p_x, p_y) &= p_x p_y, \end{aligned}$$

and so on. Let  $\tilde{\alpha} = \alpha + (1, 0)$  and observe that  $D^{\tilde{\alpha}}g = D^{(1,0)}D^\alpha g$ . The case  $\tilde{\alpha} = \alpha + (0, 1)$  is similar. Now an easy induction argument establishes the claim.  $\square$

It follows from (24) and (28) that

$$(29) \quad D^\alpha g = e^{-1/p} \sum_{l=1}^{|\alpha|} \frac{P_l(p)}{Q_l(p)} T_{\alpha,l}(p_x, p_y, \dots, D^\beta p),$$

where  $P_l, Q_l$  are polynomial functions in  $p$ .

**Lemma 4.4.** *There are constants  $C_\beta > 0$  depending on  $(0, 0) \neq \beta \in \mathbb{N} \times \mathbb{N}$  such that*

$$|D^\beta p(x, y)| \leq \frac{C_\beta}{y^{|\beta|}}, \forall (x, y) \in W_0 \setminus F_\infty.$$

*Proof.* Given  $(x, y) \in W_0 \setminus F_\infty$ , let  $n(x, y) \in \mathbb{N}$  be the unique positive integer such that  $2^{n(x,y)}(x, y) \in F_0 = \mathbb{R} \times (1/2, 1]$ . From (27), we have

$$D^\beta p(x, y) = 2^{n(x,y)|\beta|-1} D^\beta p(2^{n(x,y)}x, 2^{n(x,y)}y).$$

Let

$$0 < C_\beta := \sup_{(x,y) \in F_0} D^\beta p(x, y) < \infty.$$

It follows from the definition of  $n(x, y)$  that

$$|D^\beta p(x, y)| \leq 2^{n(x,y)|\beta|-1} C_\beta.$$

Now since  $2^{n(x,y)} \leq \frac{1}{y} \Rightarrow 2^{n(x,y)|\beta|-1} \leq \frac{1}{y^{|\beta|}}$ , the claim follows.  $\square$

**Lemma 4.5.**  $|D^\beta g(x, y)| \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ ,  $\forall \beta$ .

*Proof.* From (26) it suffices to consider  $(x, y) \in W_0 \setminus F_\infty$ . From Lemmas 4.3 and 4.4 we find constants  $C_{\alpha,l}, n_{\alpha,l} > 0$  so that

$$(30) \quad |T_{\alpha,l}(p_x, p_y, \dots, D^\beta p)| \leq \frac{C_{\alpha,l}}{y^{n_{\alpha,l}}}.$$

We can also find constants  $K_l, m_l > 0$  so that

$$(31) \quad \left| \frac{P_l(p)}{Q_l(p)} \right| \leq \frac{K_l}{p^{m_l}}.$$

Now since  $0 < y/2 \leq p(x, y) \leq 2y$ , we get from (29), (30) and (31), that

$$(32) \quad \begin{aligned} |D^\beta g(x, y)| &\leq e^{-\frac{1}{2y}} \sum_{l=1}^{|\beta|} \frac{2^{m_l} K_l C_{\alpha,l}}{y^{m_l + n_{\alpha,l}}} \\ &\leq e^{-\frac{1}{2y}} \frac{M_\beta}{y^{m_\beta}} \rightarrow 0, \end{aligned}$$

as  $y \rightarrow 0$ , where  $M_\beta, m_\beta > 0$  are suitable constants.  $\square$

Let  $\beta = (b_1, 0)$ , where  $b_1 \in \mathbb{N}$ . Since  $g|_{F_\infty \setminus \{(0,0)\}} = 0$ , we have  $D^\beta g(0, 0) = 0$ . From (32),  $D^\beta g$  is continuous at  $(0, 0)$ .

Now assume  $b_2 > 0$  and let  $\beta = (b_1, b_2)$ . Then

$$D^\beta g(0, 0) = \lim_{y \rightarrow 0^+} \frac{D^{\beta-(0,1)} g(0, y) - D^{\beta-(0,1)} g(0, 0)}{y}.$$

Using induction on  $b_2$  and inequality (32) again, we find

$$(33) \quad D^\beta g(0, 0) = 0, \forall \beta.$$

Finally from (32) and (33) we have that  $D^\beta g$  is continuous at  $(0, 0)$ . The proof of (i) is finished.

It is clear from the considerations above that  $\text{Crit}(g) \supseteq F_\infty$ . Since  $\psi$  is a local diffeomorphism in  $W_0 \setminus F_\infty$ ,  $p_2$  is a submersion and  $h'(y) > 0, \forall y > 0$ , we get that also  $g$  is a submersion when restricted to  $W_0 \setminus F_\infty$ . This implies that  $\text{Crit}(g) \subseteq F_\infty$  and, therefore,  $\text{Crit}(g) = F_\infty = \mathbb{R} \times \{0\}$ . This proves (ii).

Since  $\psi$  is the identity map near  $\partial_k^+, \forall k$ , we have that  $g(x, y) = h(y)$  for all  $(x, y)$  near  $\partial_k^+$ . This implies that

$$\nabla g(x, y) = (0, h'(1/2^k)), \forall (x, y) \in \partial_k^+.$$

Since  $h'(1/2^k) > 0, \forall k$  and  $\lim_{k \rightarrow \infty} h'(1/2^k) = 0$ , (iii) follows.

To prove (iv), observe from (21) that  $g(x, y) = h(p_2(2p_k - (x, y))) = h(3/2^{k+1} - y)$  for all  $(x, y) \in B_{p_k}(1/2^{k+4})$ . This implies in particular that

$$\nabla g(p_k) = (0, -h'(3/2^{k+2})).$$

Since  $h'(3/2^{k+2}) > 0, \forall k$  and  $\lim_{k \rightarrow \infty} h'(3/2^{k+2}) = 0$ , (iv) follows. The proof of Proposition 4.2 is now complete.

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## REFERENCES

- [1] M. Brown, *A new proof of Brouwer's lemma on translation arcs*, Houston Journal of Mathematics, **10** (1984), 35–41.
- [2] L. Guillou, *private communication*.
- [3] B. Kerékjártó, *The plane translation theorem of Brouwer and the last geometric theorem of Poincaré*. Acta Sci. Math. Szeged, **4** (1928), 86–102.
- [4] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematical Studies, **61**, Princeton University Press, Princeton, NJ, 1968.

(Salvador Addas-Zanata) DEPARTAMENTO DE MATEMÁTICA APLICADA, INSTITUTO  
DE MATEMÁTICA E ESTATÍSTICA - UNIVERSIDADE DE SÃO PAULO  
*E-mail address*, Salvador Addas-Zanata: **sazanata@ime.usp.br**

(Pedro A. S. Salomão) DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE MATEMÁ-  
TICA E ESTATÍSTICA - UNIVERSIDADE DE SÃO PAULO  
*E-mail address*, Pedro A. S. Salomão: **psalomao@ime.usp.br**